Holding versus Minimum Temperature strategies energy comparison

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Problem definition

The problem that is considered in this report is to evaluate the energies used in the following two different strategies for flat temperature management during medium-long period of disuse from residents:

1. "Holding" strategy: let the apartment cool down to MELT, Most Efficient Low Temperature, then hold the MELT for a time period and then heat it again to the TCL, Temperature Comfort Level.
2. "Minimum temperature" strategy: let the apartment cool down to the minimum reachable temperature, lowest than MELT, and then heat it to the TCL.

In order to solve the problem, the differential equation related to the temperature time behaviour is considered:

$$C\frac{dT(t)}{dt}=P_u(t)-K\left[T(t)-T_{out}(t)\right] \tag{1}$$

where $T(t)$ in K is the flat temperature, $P_u(t)$ in W the heating power, $T_{out}(t)$ in K the external temperature, C in W s / K the thermal capacitance and K in W / K the thermal conductance. A complete list of symbols can be found in Appendix 5.

Considering $P_u(t)$ and $T_{out}(t)$ constant, or better as step functions, the time behaviour of the temperature is (see Appendix 1):

$$T(t)=e^{-\frac{K}{C}(t-t_0)}T_0+\left[1-e^{-\frac{K}{C}(t-t_0)}\right]T_{out}+\left[1-e^{-\frac{K}{C}(t-t_0)}\right]\frac{P_u}{K} \quad \text{for} \quad t\geqslant t_0 \tag{2}$$

where t_0 is the initial time instant and T_0 is the initial temperature $T(t_0)$.

If a final temperature $T_f = T(t_f)$ is given, corresponding to a final time instant t_f, the time interval $\Delta t=t_f-t_0$ can be computed from (2) as (see Appendix 2):

$$\Delta t=t_f-t_0=\frac{C}{K}\ln\frac{T_0-T_{out}-\frac{P_u}{K}}{T_f-T_{out}-\frac{P_u}{K}} \tag{3}$$

Note that C/K is the system time constant, thus any time interval (3) can be seen as a multiple of this quantity.

Holding strategy

The cooling time interval is, for the "Holding" strategy:

$$\Delta t_{cool}=\frac{C}{K}\ln\frac{T_{TCL}-T_{out}}{T_{MELT}-T_{out}} \tag{4}$$

because P_u is null, $T_0=T_{TCL}$ and $T_f=T_{MELT}$.

While the heating time interval is:

$$\Delta t_{heat}=\frac{C}{K}\ln\frac{T_{MELT}-T_{out}-\frac{P_u}{K}}{T_{TCL}-T_{out}-\frac{P_u}{K}} \tag{5}$$

because initial and final temperatures exchange each other.
The holding time interval can be computed simply as a difference, if the considered total interval time horizon Δt_{hor} is given:

$$\Delta t_{hold} = \Delta t_{hor} - \Delta t_{cool} - \Delta t_{heat} \tag{6}$$

To compute the energies given by the heating device it is necessary to compute the used power for the holding and heating time intervals (of course the lost energy during cooling is of no interest).
During holding interval the power is equal to the heat dispersion towards the external environment:

$$P_{hold} = K\left(T_{MELT} - T_{out}\right) \tag{7}$$

and so the associated energy is:

$$E_{hold}^{HO} = K\left(T_{MELT} - T_{out}\right) \cdot \Delta t_{hold} \tag{8}$$

While during heating interval the power is simply equal to P_u:

$$P_{heat} = P_u \tag{9}$$

and so the associated energy is:

$$E_{heat}^{HO} = P_u \cdot \Delta t_{heat} \tag{10}$$

Then the total energy is

$$E^{HO} = E_{hold}^{HO} + E_{heat}^{HO} = K\left(T_{MELT} - T_{out}\right) \cdot \Delta t_{hold} + P_u \cdot \Delta t_{heat} \tag{11}$$

Minimum temperature strategy

The cooling time interval is, for the "Minimum Temperature" strategy:

$$\Delta t_{cool} = \frac{C}{K} \ln \frac{T_{TCL} - T_{out}}{T_{min} - T_{out}} \tag{12}$$

While the heating time interval is:

$$\Delta t_{heat} = \frac{C}{K} \ln \frac{T_{min} - T_{out} - \frac{P_u}{K}}{T_{TCL} - T_{out} - \frac{P_u}{K}} \tag{13}$$

The minimum temperature T_{min} in (12) and (13) can be computed considering that the sum of the two time intervals must be equal to the total time horizon:

$$\Delta t_{hor} = \Delta t_{cool} + \Delta t_{heat} \tag{14}$$

Substituting (12) and (13) in (14) and solving for T_{min} gives (see Appendix 3):

$$T_{min} = T_{out} + \frac{P_u\left(T_{TCL} - T_{out}\right)}{\left[1 - e^{\frac{K}{C}\Delta t_{hor}}\right] K\left(T_{TCL} - T_{out}\right) + P_u e^{\frac{K}{C}\Delta t_{hor}}} \tag{15}$$

> **Note**: this minimum temperature is a lower bound for the MELT temperature of the "Holding" strategy, in the sense that, if a MELT temperature less than this minimum temperature is chosen, the sum of cooling, holding and heating time intervals is greater than the total time horizon or that, considering (6), a negative holding interval is obtained.

To compute the energy for this strategy only the heating interval must be considered, with the associated power $P_{heat}=P_u$. Thus the energy is:

$$E^{MT} = E_{heat}^{MT} = P_u \cdot \Delta t_{heat} \tag{16}$$

Energies comparison and simulations

To decide which one of the two strategies is better, in the sense of using a minimum amount of energy, we need to compare

$$E^{HO} = E_{hold}^{HO} + E_{heat}^{HO} \quad \text{with} \quad E^{MT} = E_{heat}^{MT} \tag{17}$$

and select the strategy with minimum energy.
By simulation, considering the variation of each one of the possible parameters and variables, it was found that the "Minimum Temperature" is the best strategy, in the sense that:

$$E^{MT} < E^{HO} \tag{18}$$

An example is drawn in Figure 1, where the time behaviours of the temperature and the power are depicted for both the strategies (HO solid lines, MT dashed lines), obtained considering the numerical values given in Table 1.

K = 55 W/K	C = 9˙450˙000 Ws/K	
P_u = 2800 W	Δt_{hor} = 259˙200 s (3 days)	
T_{TCL} = 21°C	T_{out} = 2°C	T_{MELT} = 12°C

Table 1: Simulation numerical values.

The computed values are: T_{min} = 7,9255°C, E^{MT} = 45,8915 kWh and E^{HO} = 49,4326 kWh. Thus the "Minimum Temperature" strategy saves $E^{HO}-E^{MT}$ = 3,5411 kWh of energy. Graphically the energies are the areas under the power horizontal lines. In dash-dotted lines the *TCL* and *out* temperatures.

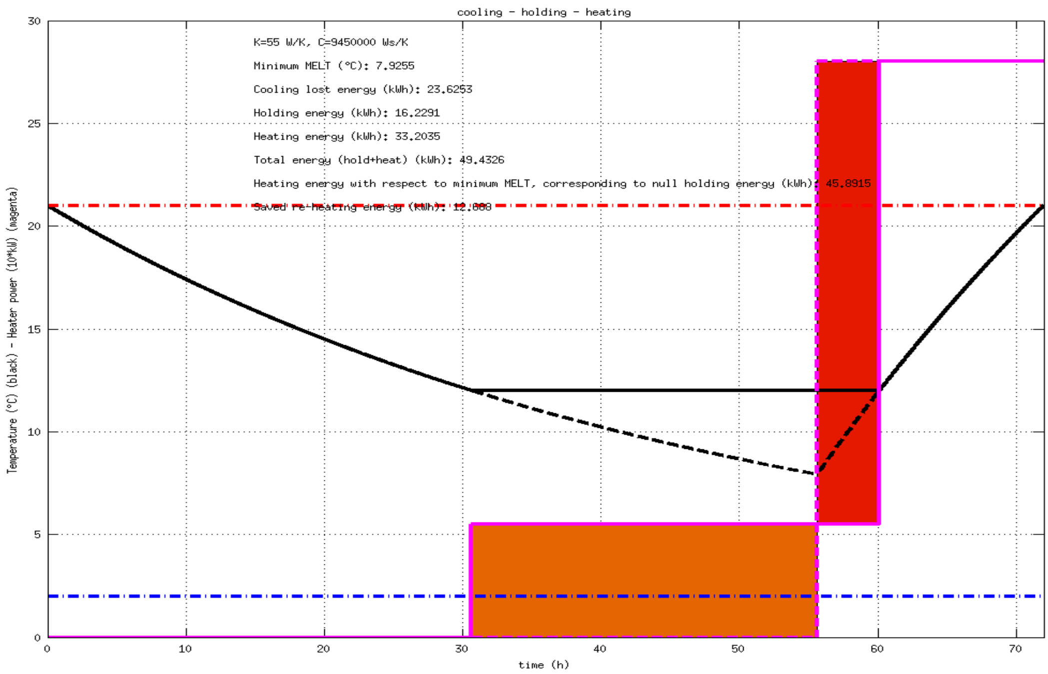

Figure 1: Simulation example: Holding strategy solid, Minimum Temperature strategy dashed.

In order to compare the two strategies the differences between the two areas can be considered and these are represented in Figure 1 by the two coloured areas: the red area is smaller than the orange one.
To completely demonstrate that the MT strategy is always better than the HO strategy, the analytical proof of the inequality $E^{MT} < E^{HO}$, for all the possible parameters and variables numerical values, has to be obtained.
Before presenting this proof, it is convenient to solve the problem, even yet numerically, also as a minimization problem.

Energy optimization problem

Consider again the total energy for the Holding strategy as a function of the MELT temperature:

$$E^{HO}(T_{MELT}) = E_{hold}^{HO}(T_{MELT}) + E_{heat}^{HO}(T_{MELT}) \tag{19}$$

and minimize it, that is to say, solve the optimization problem

$$\min_{T_{MELT}} E^{HO}(T_{MELT}) \quad \text{subject to} \quad \Delta t_{hold} \geqslant 0 \tag{20}$$

The constraint in (20) is equivalent to impose $T_{MELT} \geq T_{min}$.
Using, for example, the Matlab Optimization tool, the obtained solution is

$$\arg\min_{T_{MELT}} E^{HO}(T_{MELT}) = T_{min} \tag{21}$$

that is to say that the minimum energy consumption is obtained using the “Minimum Temperature” strategy.
The behaviour of $E^{HO}(T_{MELT})$, using the data from Table 1, is depicted in Figure 2. It can be seen that the total energy is a monotonically increasing function in the temperature interval of interest [T_{min} T_{TCL}] and the minimum is obtained on the left extreme of this interval: T_{min} = 7,9255°C. The minimum corresponds to a null holding energy and the best strategy is again the “MinimumTemperature” one.

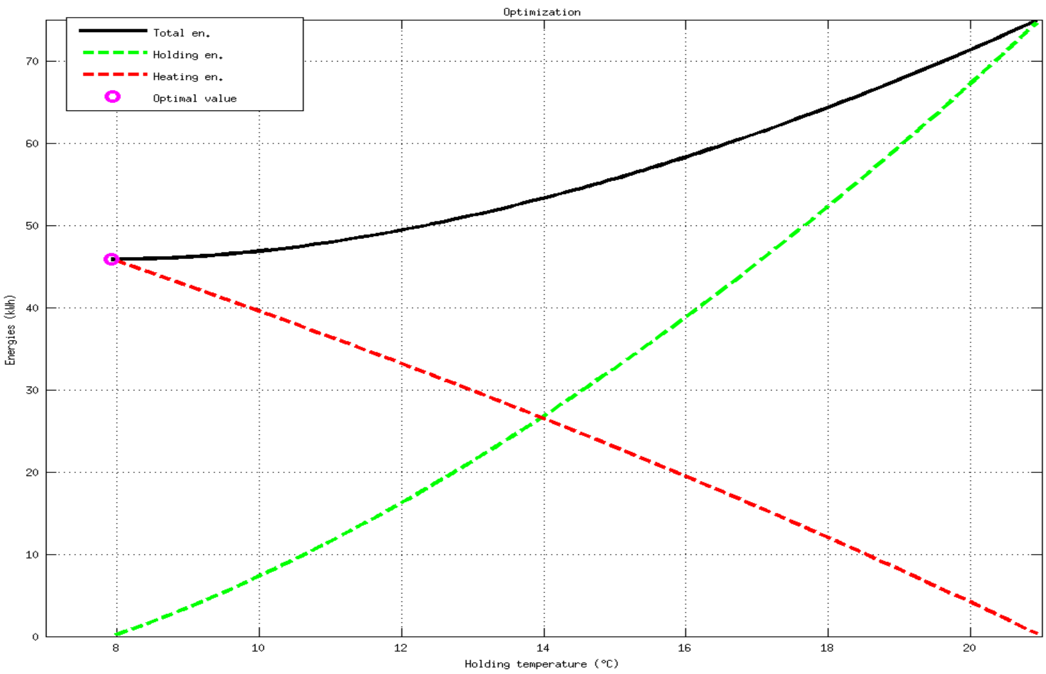

Figure 2: Energies behaviours.

Note that the energy curve tends to an horizontal line and, from (15), the minimum temperature tends to the TCL one, if the heating power P_u is decreased. In general power P_u must satisfy the inequality

$$P_u > K(T_{MELT} - T_{out}) \tag{22}$$

in order to maintain the TCL temperature. If this lower bound is reached, no temperature greater than TCL can be obtained.
For example, using data in Table 1, it is obtained P_u > 1045 W and setting P_u = 1050 W the result

depicted in Figure 3 is obtained.

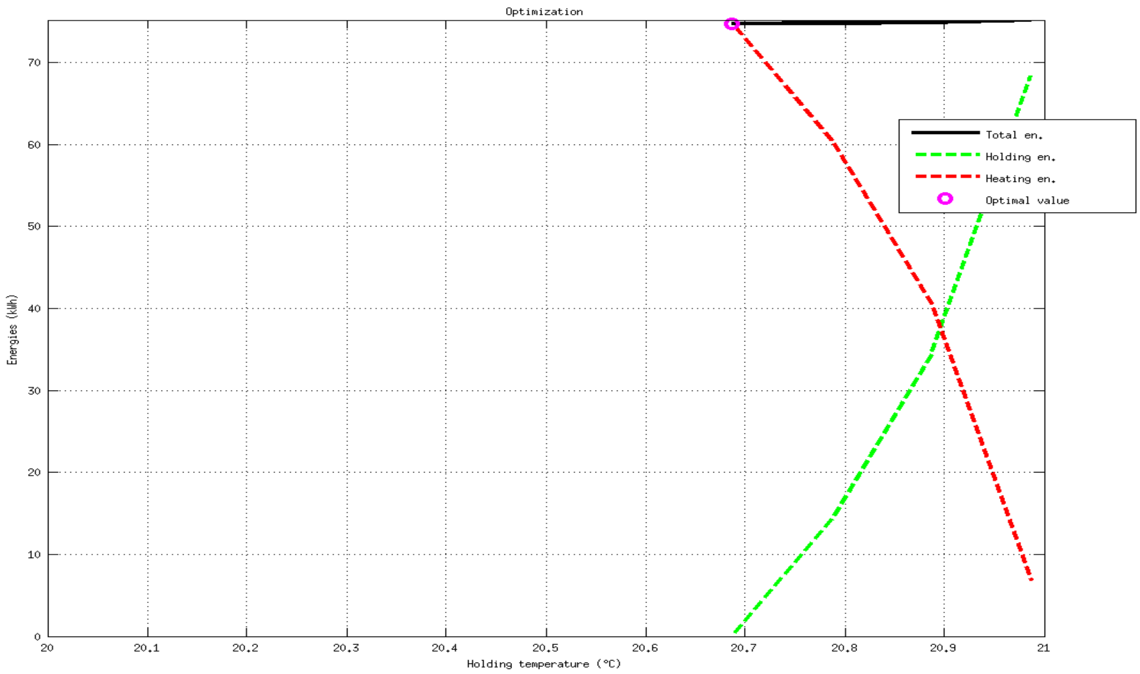

Figure 3: Energies behaviours with P_u close to its lower bound.

Minimum energy analytical proof

The proof of the optimal solution for the energy consumption can be obtained by demonstrating that the energy function is continuous and monotonically increasing, that is to say that its first derivative is always positive, on the interest interval [T_{min} T_{TCL}].
The continuity on the interval [T_{min} T_{TCL}], with respect to T_{MELT}, is checked by direct inspection of the energy function

$$E^{HO}(T_{MELT})= \\ =K\left(T_{MELT}-T_{out}\right)\cdot\left[\Delta t_{hor}-\frac{C}{K}\ln\frac{T_{TCL}-T_{out}}{T_{MELT}-T_{out}}-\frac{C}{K}\ln\frac{T_{MELT}-T_{out}-\frac{P_u}{K}}{T_{TCL}-T_{out}-\frac{P_u}{K}}\right]+ \\ +P_u\cdot\frac{C}{K}\ln\frac{T_{MELT}-T_{out}-\frac{P_u}{K}}{T_{TCL}-T_{out}-\frac{P_u}{K}} \quad (23)$$

that is given by sums and multiplication of continuous functions.
The energy first derivative with respect to T_{MELT} is (see Appendix 4):

$$\frac{dE^{HO}}{dT_{MELT}}=K\,\Delta t_{hor}+C\ln\left[\frac{\frac{P_u}{K(T_{TCL}-T_{out})}-1}{\frac{P_u}{K(T_{MELT}-T_{out})}-1}\right] \quad (24)$$

and its positivity holds if

$$\frac{dE^{HO}}{dT_{MELT}}\geqslant 0 \;\Rightarrow\; K\,\Delta t_{hor}\geqslant C\ln\left[\frac{\frac{P_u}{K(T_{MELT}-T_{out})}-1}{\frac{P_u}{K(T_{TCL}-T_{out})}-1}\right] \;\Rightarrow\; e^{\frac{K}{C}\Delta t_{hor}}\geqslant\frac{\frac{P_u}{K(T_{MELT}-T_{out})}-1}{\frac{P_u}{K(T_{TCL}-T_{out})}-1} \quad (25)$$

Now, referring to (46) in Appendix 3, the left hand side of the last inequality in (25) is

$$e^{\frac{K}{C}\Delta t_{hor}}=\frac{\frac{P_u}{K(T_{min}-T_{out})}-1}{\frac{P_u}{K(T_{TCL}-T_{out})}-1}\geqslant\frac{\frac{P_u}{K(T_{MELT}-T_{out})}-1}{\frac{P_u}{K(T_{TCL}-T_{out})}-1} \tag{26}$$

Then, simplifying the common terms, the inequality becomes

$$\frac{1}{T_{min}-T_{out}}\geqslant\frac{1}{T_{MELT}-T_{out}} \tag{27}$$

$$T_{min}-T_{out}\leqslant T_{MELT}-T_{out} \tag{28}$$

This inequality is always verified due to the fact that T_{min} is a lower bound for T_{MELT}.
This demonstrate that the energy is a continuous monotonically increasing function and thus that its minimum is reached on the left extreme of the interval [T_{min} T_{TCL}], that is to say for $T_{MELT} = T_{min}$: the "Minimum Temperature" strategy. Note also that the derivative is zero only when $T_{MELT} = T_{min}$, for all the other values of T_{MELT} it is strictly positive.

Conclusions

It can be stated that both numerical and analytical studies indicate that there is no convenience in holding a MELT temperature because the requested energy is always greater than the one used to heat the apartment from the minimum reachable temperature.

Appendix 1 Differential equation solution

Consider the differential equation (2) and Laplace transform it:

$$C\left[sT(s)-T_0\right]=P_u(s)-K\left[T(s)-T_{out}(s)\right] \tag{29}$$

remember that both P_u and T_{out} are supposed to be step functions, then:

$$C\left[sT(s)-T_0\right]=\frac{P_u}{s}-KT(s)+K\frac{T_{out}}{s} \tag{30}$$

$$sT(s)+\frac{K}{C}T(s)=T_0+K\frac{T_{out}}{Cs}+\frac{P_u}{Cs} \tag{31}$$

$$\left(s+\frac{K}{C}\right)T(s)=T_0+\frac{K}{Cs}T_{out}+\frac{1}{Cs}P_u \tag{32}$$

$$T(s)=\frac{1}{s+\frac{K}{C}}T_0+\frac{1}{s\left(s+\frac{K}{C}\right)}\left(\frac{K}{C}T_{out}+\frac{1}{C}P_u\right) \tag{33}$$

$$T(s)=\frac{1}{s+\frac{K}{C}}T_0+\frac{C}{K}\left(\frac{1}{s}-\frac{1}{s+\frac{K}{C}}\right)\left(\frac{K}{C}T_{out}+\frac{1}{C}P_u\right) \tag{34}$$

$$T(s)=\frac{1}{s+\frac{K}{C}}T_0+\left(\frac{1}{s}-\frac{1}{s+\frac{K}{C}}\right)\left(T_{out}+\frac{P_u}{K}\right) \tag{35}$$

$$T(t)=e^{-\frac{K}{C}(t-t_0)}T_0+\left[1-e^{-\frac{K}{C}(t-t_0)}\right]\left(T_{out}+\frac{P_u}{K}\right) \tag{36}$$

Appendix 2 Time interval computation

Substitute in (2) t_f in t and T_f in $T(t)$ and obtain

$$T_f=e^{-\frac{K}{C}(t_f-t_0)}T_0+\left[1-e^{-\frac{K}{C}(t_f-t_0)}\right]T_{out}+\left[1-e^{-\frac{K}{C}(t_f-t_0)}\right]\frac{P_u}{K} \tag{37}$$

Call $\Delta t = t_f - t_0$ and factorize

$$T_f=\left[T_0-T_{out}-\frac{P_u}{K}\right]e^{-\frac{K}{C}\Delta t}+T_{out}+\frac{P_u}{K} \tag{38}$$

then

$$T_f - T_{out} - \frac{P_u}{K} = \left[T_0 - T_{out} - \frac{P_u}{K} \right] e^{-\frac{K}{C}\Delta t} \tag{39}$$

$$\frac{T_f - T_{out} - \frac{P_u}{K}}{T_0 - T_{out} - \frac{P_u}{K}} = e^{-\frac{K}{C}\Delta t} \tag{40}$$

$$\ln \frac{T_f - T_{out} - \frac{P_u}{K}}{T_0 - T_{out} - \frac{P_u}{K}} = -\frac{K}{C}\Delta t \tag{41}$$

$$\Delta t = -\frac{C}{K} \ln \frac{T_f - T_{out} - \frac{P_u}{K}}{T_0 - T_{out} - \frac{P_u}{K}} \tag{42}$$

$$\Delta t = \frac{C}{K} \ln \frac{T_0 - T_{out} - \frac{P_u}{K}}{T_f - T_{out} - \frac{P_u}{K}} \tag{43}$$

Appendix 3 Minimum temperature computation

Substitute (12) and (13) in (14) and obtain

$$\begin{aligned} \Delta t_{hor} &= \Delta t_{cool} + \Delta t_{heat} = \frac{C}{K} \ln \frac{T_{TCL} - T_{out}}{T_{min} - T_{out}} + \frac{C}{K} \ln \frac{T_{min} - T_{out} - \frac{P_u}{K}}{T_{TCL} - T_{out} - \frac{P_u}{K}} = \\ &= \frac{C}{K} \ln \left(\frac{T_{TCL} - T_{out}}{T_{min} - T_{out}} \cdot \frac{T_{min} - T_{out} - \frac{P_u}{K}}{T_{TCL} - T_{out} - \frac{P_u}{K}} \right) = \frac{C}{K} \ln \left(\frac{T_{TCL} - T_{out}}{T_{min} - T_{out}} \cdot \frac{T_{min} - T_{out}}{T_{TCL} - T_{out}} \cdot \frac{1 - \frac{P_u}{K(T_{min} - T_{out})}}{1 - \frac{P_u}{K(T_{TCL} - T_{out})}} \right) = \\ &= \frac{C}{K} \ln \left(\frac{1 - \frac{P_u}{K(T_{min} - T_{out})}}{1 - \frac{P_u}{K(T_{TCL} - T_{out})}} \right) \end{aligned} \tag{44}$$

then taking the first and last sides of the equality

$$\frac{K}{C} \Delta t_{hor} = \ln \left(\frac{1 - \frac{P_u}{K(T_{min} - T_{out})}}{1 - \frac{P_u}{K(T_{TCL} - T_{out})}} \right) \tag{45}$$

$$e^{\frac{K}{C}\Delta t_{hor}} = \frac{1 - \frac{P_u}{K(T_{min} - T_{out})}}{1 - \frac{P_u}{K(T_{TCL} - T_{out})}} \tag{46}$$

$$\left[1 - \frac{P_u}{K(T_{TCL} - T_{out})} \right] e^{\frac{K}{C}\Delta t_{hor}} = 1 - \frac{P_u}{K(T_{min} - T_{out})} \tag{47}$$

$$\frac{P_u}{K(T_{min} - T_{out})} = 1 - \left[1 - \frac{P_u}{K(T_{TCL} - T_{out})} \right] e^{\frac{K}{C}\Delta t_{hor}} \tag{48}$$

$$\frac{P_u}{K(T_{min}-T_{out})}=\frac{K(T_{TCL}-T_{out})-[K(T_{TCL}-T_{out})-P_u]e^{\frac{K}{C}\Delta t_{hor}}}{K(T_{TCL}-T_{out})} \quad (49)$$

$$T_{min}-T_{out}=\frac{P_u(T_{TCL}-T_{out})}{K(T_{TCL}-T_{out})-[K(T_{TCL}-T_{out})-P_u]e^{\frac{K}{C}\Delta t_{hor}}} \quad (50)$$

$$T_{min}=T_{out}+\frac{P_u(T_{TCL}-T_{out})}{\left[1-e^{\frac{K}{C}\Delta t_{hor}}\right]K(T_{TCL}-T_{out})+P_u e^{\frac{K}{C}\Delta t_{hor}}} \quad (51)$$

Appendix 4 Energy first derivative

Consider equation (23)

$$\begin{aligned}
&E^{HO}(T_{MELT})=\\
&=K\left(T_{MELT}-T_{out}\right)\cdot\left[\Delta t_{hor}-\frac{C}{K}\ln\frac{T_{TCL}-T_{out}}{T_{MELT}-T_{out}}-\frac{C}{K}\ln\frac{T_{MELT}-T_{out}-\frac{P_u}{K}}{T_{TCL}-T_{out}-\frac{P_u}{K}}\right]+\\
&+P_u\cdot\frac{C}{K}\ln\frac{T_{MELT}-T_{out}-\frac{P_u}{K}}{T_{TCL}-T_{out}-\frac{P_u}{K}}=\\
&=K\left(T_{MELT}-T_{out}\right)\cdot\Delta t_{hor}+\\
&+C\left(T_{MELT}-T_{out}\right)\ln\left[\frac{P_u-K(T_{TCL}-T_{out})}{P_u-K(T_{MELT}-T_{out})}\cdot\frac{T_{MELT}-T_{out}}{T_{TCL}-T_{out}}\right]+\\
&+P_u\cdot\frac{C}{K}\ln\frac{P_u-K(T_{MELT}-T_{out})}{P_U-K(T_{TCL}-T_{out})}
\end{aligned} \quad (52)$$

and compute the derivative with respect to T_{MELT}

$$\begin{aligned}
&\frac{d\,E^{HO}(T_{MELT})}{d\,T_{MELT}}=K\,\Delta t_{hor}+C\ln\left[\frac{P_u-K(T_{TCL}-T_{out})}{P_u-K(T_{MELT}-T_{out})}\cdot\frac{T_{MELT}-T_{out}}{T_{TCL}-T_{out}}\right]+\\
&+C\left(T_{MELT}-T_{out}\right)\frac{P_u-K(T_{MELT}-T_{out})}{P_u-K(T_{TCL}-T_{out})}\cdot\frac{T_{TCL}-T_{out}}{T_{MELT}-T_{out}}\cdot\\
&\cdot\frac{[P_u-K(T_{TCL}-T_{out})]\cdot[P_u-K(T_{MELT}-T_{out})]\cdot(T_{TCL}-T_{out})+[P_u-K(T_{TCL}-T_{out})]\cdot(T_{MELT}-T_{out})\cdot K(T_{TCL}-T_{out})}{\{[P_u-K(T_{MELT}-T_{out})]\cdot(T_{TCL}-T_{out})\}^2}+\\
&+P_u\frac{C}{K}\cdot\frac{P_U-K(T_{TCL}-T_{out})}{P_u-K(T_{MELT}-T_{out})}\cdot\frac{-K}{P_U-K(T_{TCL}-T_{out})}=\\
&=K\,\Delta t_{hor}+C\ln\frac{\frac{P_u}{K(T_{TCL}-T_{out})}-1}{\frac{P_u}{K(T_{MELT}-T_{out})}-1}+C\cdot\frac{P_u}{\{P_u-K(T_{MELT}-T_{out})\}}-C\frac{P_u}{P_u-K(T_{MELT}-T_{out})}=\\
&=K\,\Delta t_{hor}+C\ln\frac{\frac{P_u}{K(T_{TCL}-T_{out})}-1}{\frac{P_u}{K(T_{MELT}-T_{out})}-1}
\end{aligned} \quad (53)$$

Appendix 5 List of symbols

Symbol	Unit	Definition
C	W s / K	Thermal capacitance
K	W / K	Thermal conductance
T	K	Temperature
T_0	K	Initial temperature
T_f	K	Final temperature
T_{out}	K	External temperature
T_{TCL}	K	Temperature Comfort Level
T_{MELT}	K	Most Efficient Low Temperature
T_{min}	K	Minimum reachable temperature
P	W	Power
P_u	W	Heating device power
E	Ws	Energy
t	s	Time variable
t_0	s	Initial time
t_f	s	Final time
Δt	s	Time interval
Δt_{hor}	s	Total time horizon interval